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Thermocapillary flows in a three-layer system with a temperature gradient along the interfaces

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Abstract

The nonlinear stability of three superposed liquid layers bounded by two solid planes and subjected to a temperature gradient, directed along the interfaces, is investigated. The periodic boundary conditions on the lateral walls are considered. The nonlinear simulations of the wavy convective regimes are performed by the finite-difference method.

Keywords: Thermocapillary flows; Instabilities; Interfaces

1. Introduction

Stability of convective flows in systems with an interface has been a subject of an extensive investigation. Several classes of instabilities have been found by means of the linear stability theory for purely thermocapillary flows [1–4] and for buoyant-thermocapillary flows [5–8]. For the most typical kind of instability, hydrothermal instability, the appearance of oblique waves moving upstream has been predicted by the theory and justified in experiments [9–11].

The most of the investigations have been fulfilled for a sole liquid layer with a free surface, i.e., in the framework of the one-layer approach. Recently, Madruga et al. [12,13] studied the linear stability of two superposed horizontal liquid layers bounded by two solid planes and subjected to a horizontal temperature gradient. The nonlinear wavy convective regimes in two-layer systems were described in [14].

Numerical investigations of thermocapillary convection in multilayer systems were started in [15–17]. In these papers the linear stability of the mechanical equilibrium state and the nonlinear regimes of convection have been studied. Prakash and Koster found the analytical solution describing the velocity and the temperature fields for the parallel flow in the core region of a three-layer fluid system under the action of the temperature gradient directed along the interfaces (see [18,19]). The flow field in the end-wall region was analyzed by matching with the core region flow [19,20]. The nonlinear simulations of convective flows in a closed cavity filled by a symmetric three-layer system, have been performed in [21] (for a review, see [22]).

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In the present paper, we investigate the nonlinear stability of the three-layer systems with *periodic boundary conditions* on lateral boundaries, subjected to a temperature gradient directed along the interfaces. The nonlinear simulations of the wavy convective regimes in these systems were never studied before.

The paper is organized as follows. In Section 2, the mathematical formulation of the problem is presented. The nonlinear approach is described in Section 3. Nonlinear simulations of wavy convective flows in infinite layers are considered in Section 4. Section 5 contains some concluding remarks.

2. General equations and boundary conditions

We consider a system of three horizontal layers of immiscible viscous fluids with different physical properties (see Fig. 1). The thicknesses of the layers are a_m, $m = 1, 2, 3$. The m-th fluid has density ρ_m, kinematic viscosity ν_m, dynamic viscosity $\eta_m = \rho_m \nu_m$, thermal diffusivity χ_m, and heat conductivity κ_m. The system is bounded from above and from below by two rigid plates, $z = a_1$ and $z = -a_2 - a_3$. A constant temperature gradient is imposed in the direction of the axis x: $T_1(x, y, a_1, t) = T_3(x, y, -a_2 - a_3, t) = -Ax + const$, $A > 0$. The surface tension coefficients on the upper and lower interfaces, σ and σ_*, are linear functions of temperature T: $\sigma = \sigma_0 - \alpha T$, $\sigma_* = \sigma_{*0} - \alpha_* T$, where $\alpha > 0$ and $\alpha_* > 0$.

Let us define

$$\rho = \frac{\rho_1}{\rho_2}, \quad \nu = \frac{\nu_1}{\nu_2}, \quad \eta = \frac{\eta_1}{\eta_2} = \rho\nu, \quad \chi = \frac{\chi_1}{\chi_2}, \quad \kappa = \frac{\kappa_1}{\kappa_2}, \quad a = \frac{a_2}{a_1},$$
$$\rho_* = \frac{\rho_1}{\rho_3}, \quad \nu_* = \frac{\nu_1}{\nu_3}, \quad \eta_* = \frac{\eta_1}{\eta_3} = \rho_*\nu_*, \quad \chi_* = \frac{\chi_1}{\chi_3}, \quad \kappa_* = \frac{\kappa_1}{\kappa_3}, \quad a_* = \frac{a_3}{a_1}, \quad \bar{\alpha} = \frac{\alpha_*}{\alpha}.$$

As the units of length, time, velocity, pressure and temperature we use a_1, a_1^2/ν_1, ν_1/a_1, $\rho_1\nu_1^2/a_1^2$ and Aa_1. The complete nonlinear equations governing convection are then written in the following dimensionless form:

$$\frac{\partial \mathbf{v}_m}{\partial t} + (\mathbf{v}_m \cdot \nabla)\mathbf{v}_m = -e_m \nabla p_m + c_m \Delta \mathbf{v}_m, \tag{1}$$
$$\frac{\partial T_m}{\partial t} + \mathbf{v}_m \cdot \nabla T_m = \frac{d_m}{P} \Delta T_m, \tag{2}$$
$$\nabla \mathbf{v}_m = 0, \quad m = 1, 2, 3, \tag{3}$$

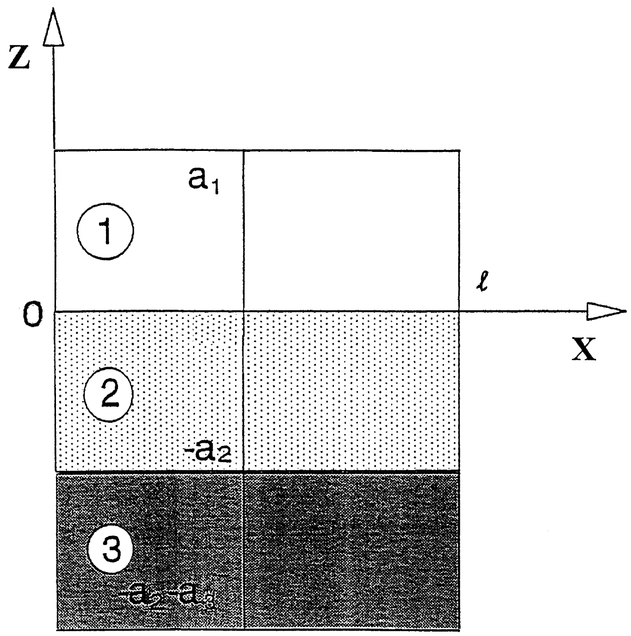

Fig. 1. Geometrical configuration of the three-layer system and coordinate axes.

where $e_1 = c_1 = d_1 = 1$, $e_2 = \rho$, $c_2 = 1/\nu$, $d_2 = 1/\chi$, $e_3 = \rho_*$, $c_3 = 1/\nu_*$, $d_3 = 1/\chi_*$; $\Delta = \nabla^2$, and $P = \nu_1/\chi_1$ is the Prandtl number.

The boundary conditions on the isothermic rigid boundaries are:

$$\mathbf{v}_1 = 0, \quad T_1 = T_0 - x \quad \text{at } z = 1, \tag{4}$$

$$\mathbf{v}_3 = 0, \quad T_3 = T_0 - x \quad \text{at } z = -a - a_*, \tag{5}$$

where T_0 is constant.

We assume that the interfaces between fluids are flat and situated at $z = 0$ and $z = -a$, and put the following system of boundary conditions: at $z = 0$

$$\frac{\partial v_{1x}}{\partial z} - \eta^{-1}\frac{\partial v_{2x}}{\partial z} - \frac{M}{P}\frac{\partial T_1}{\partial x} = 0, \quad \frac{\partial v_{1y}}{\partial z} - \eta^{-1}\frac{\partial v_{2y}}{\partial z} - \frac{M}{P}\frac{\partial T_1}{\partial y} = 0, \tag{6}$$

$$v_{1x} = v_{2x}, \quad v_{1y} = v_{2y}, \quad v_{1z} = v_{2z} = 0, \tag{7}$$

$$T_1 = T_2, \tag{8}$$

$$\frac{\partial T_1}{\partial z} = \kappa^{-1}\frac{\partial T_2}{\partial z}; \tag{9}$$

at $z = -a$

$$\eta^{-1}\frac{\partial v_{2x}}{\partial z} - \eta_*^{-1}\frac{\partial v_{3x}}{\partial z} - \frac{\bar{\alpha} M}{P}\frac{\partial T_3}{\partial x} = 0, \qquad \eta^{-1}\frac{\partial v_{2y}}{\partial z} - \eta_*^{-1}\frac{\partial v_{3y}}{\partial z} - \frac{\bar{\alpha} M}{P}\frac{\partial T_3}{\partial y} = 0, \tag{10}$$

$$v_{2x} = v_{3x}, \quad v_{2y} = v_{3y}, \quad v_{2z} = v_{3z} = 0, \tag{11}$$

$$T_2 = T_3, \tag{12}$$

$$\kappa^{-1}\frac{\partial T_2}{\partial z} = \kappa_*^{-1}\frac{\partial T_3}{\partial z}. \tag{13}$$

Here $M = \alpha A a_1^2/\eta_1\chi_1$ is the Marangoni number.

3. Nonlinear approach

In order to investigate the flow regimes generated by the convective instabilities, we perform nonlinear simulations of two-dimensional flows ($v_{my} = 0$ ($m = 1, 2, 3$); the fields of physical variables do not depend on y). In this case, we can introduce the stream function ψ_m and the vorticity ϕ_m,

$$v_{m,x} = \frac{\partial\psi_m}{\partial z}, \qquad v_{m,z} = -\frac{\partial\psi_m}{\partial x},$$

$$\phi_m = \frac{\partial v_{m,z}}{\partial x} - \frac{\partial v_{m,x}}{\partial z} \quad (m = 1, 2, 3)$$

and rewrite Eqs. (1)–(3) in the following form:

$$\frac{\partial\phi_m}{\partial t} + \frac{\partial\psi_m}{\partial z}\frac{\partial\phi_m}{\partial x} - \frac{\partial\psi_m}{\partial x}\frac{\partial\phi_m}{\partial z} = c_m\Delta\phi_m, \tag{14}$$

$$\Delta\psi_m = -\phi_m, \tag{15}$$

$$\frac{\partial T_m}{\partial t} + \frac{\partial\psi_m}{\partial z}\frac{\partial T_m}{\partial x} - \frac{\partial\psi_m}{\partial x}\frac{\partial T_m}{\partial z} = \frac{d_m}{P}\Delta T_m \quad (m = 1, 2, 3). \tag{16}$$

Coefficients c_m and d_m have been defined in Section 2. At the interfaces normal components of velocity vanish and the continuity conditions for tangential components of velocity, viscous stresses, temperatures, and heat fluxes also apply:

$$z = 0: \quad \psi_1 = \psi_2 = 0, \quad \frac{\partial\psi_1}{\partial z} = \frac{\partial\psi_2}{\partial z}, \quad T_1 = T_2, \tag{17}$$

$$\frac{\partial T_1}{\partial z} = \frac{1}{\kappa}\frac{\partial T_2}{\partial z}, \qquad \frac{\partial^2\psi_1}{\partial z^2} = \frac{1}{\eta}\frac{\partial^2\psi_2}{\partial z^2} + \frac{M}{P}\frac{\partial T_1}{\partial x}, \tag{18}$$

$$z=-a:\quad \psi_2=\psi_3=0,\quad \frac{\partial\psi_2}{\partial z}=\frac{\partial\psi_3}{\partial z},\quad T_2=T_3, \tag{19}$$

$$\frac{1}{\kappa}\frac{\partial T_2}{\partial z}=\frac{1}{\kappa_*}\frac{\partial T_3}{\partial z},\qquad \frac{1}{\eta}\frac{\partial^2\psi_2}{\partial z^2}=\frac{1}{\eta_*}\frac{\partial^2\psi_3}{\partial z^2}+\frac{\bar{\alpha}M}{P}\frac{\partial T_2}{\partial x}. \tag{20}$$

On the horizontal solid plates the boundary conditions read:

$$z=1:\quad \psi_1=\frac{\partial\psi_1}{\partial z}=0,\quad T_1=T_0-x, \tag{21}$$

$$z=-a-a_*:\quad \psi_3=\frac{\partial\psi_3}{\partial z}=0,\quad T_3=T_0-x. \tag{22}$$

The transformation of variables $T_m=\Theta_m-x$, $m=1,2,3$, has been used.

For simulations of cellular motions in an infinite layers, one can use the periodic boundary conditions:

$$\psi_m(x+L,z)=\psi_m(x,z),\quad \phi_m(x+L,z)=\phi_m(x,z),\quad \Theta_m(x+L,z)=\Theta_m(x,z). \tag{23}$$

The boundary value problem formulated above was solved by the finite-difference method. Eqs. (14)–(16) were approximated on a uniform mesh using a second-order approximation for the spatial coordinates. The nonlinear equations were solved using the explicit scheme, on a rectangular uniform mesh 84×56 ($L=3.2$) and 168×56 ($L=16$). The time step was calculated by the formula

$$\Delta t=\frac{[\min(\Delta x,\Delta z)]^2[\min(1,\nu,\chi,\nu_*,\chi_*)]}{2[2+\max|\psi_m(x,z)|]},$$

where Δx, Δz are the mesh sizes for the corresponding coordinates. The Poisson equations were solved by the iterative Liebman successive over-relaxation method on each time step: the accuracy of the solution was 10^{-5}.

At the interfaces the expressions for the vorticities at the exterior layers are approximated with the second-order accuracy for the spatial coordinates and have a form

$$\phi_1(x,0)=-\frac{2[\psi_2(x,-\Delta z)+\psi_1(x,\Delta z)]}{(\Delta z)^2(1+\eta)}-\frac{M}{P}\frac{\eta}{1+\eta}\frac{\partial T_1}{\partial x}(x,0), \tag{24}$$

$$\phi_2(x,0)=\eta\phi_1(x,0)+\frac{\eta M}{P}\frac{\partial T_1}{\partial x}(x,0), \tag{25}$$

$$\phi_2(x,-a)=-\frac{2[\psi_3(x,-a-\Delta z)+\psi_2(x,-a+\Delta z)]}{(\Delta z)^2(1+\eta_*\eta^{-1})}-\frac{\bar{\alpha}M}{P}\frac{\eta_*}{1+\eta_*\eta^{-1}}\frac{\partial T_2}{\partial x}(x,-a), \tag{26}$$

$$\phi_3(x,-a)=\eta_*\eta^{-1}\phi_2(x,-a)+\frac{\eta_*\bar{\alpha}M}{P}\frac{\partial T_2}{\partial x}(x,-a). \tag{27}$$

The temperatures on the interfaces were calculated by the second-order approximation formulas:

$$T_2(x,0)=T_1(x,0)=\frac{[4T_2(x,-\Delta z)-T_2(x,-2\Delta z)]+\kappa[4T_1(x,\Delta z)-T_1(x,2\Delta z)]}{3(1+\kappa)}, \tag{28}$$

$$T_2(x,-a)=T_3(x,-a)=\frac{\kappa_*[4T_2(x,-a+\Delta z)-T_2(x,-a+2\Delta z)]+\kappa[4T_3(x,-a-\Delta z)-T_3(x,-a-2\Delta z)]}{3(\kappa+\kappa_*)}. \tag{29}$$

The details of the numerical method can be found in the book by Simanovskii and Nepomnyashchy [23].

4. Numerical results

4.1. The system 47v2 silicone oil–water–47v2 silicone oil

We shall start our analysis by the consideration of the real symmetric 47v2 silicone oil–water–47v2 silicone oil system with the following set of parameters: $\eta=1.7375$, $\nu=2$, $\kappa=0.184$, $\chi=0.778$; $\eta_*=\nu_*=\kappa_*=\chi_*=1$; $P=25.7$. We take $a=a_*=1$. It means that the exterior layers have the same thermophysical properties. For any values of the Marangoni numbers M ($M\neq 0$), the boundary value problem has a solution $\psi_m=\psi_m^0(z)$, $\Theta_m=\Theta_m^0(z)$,

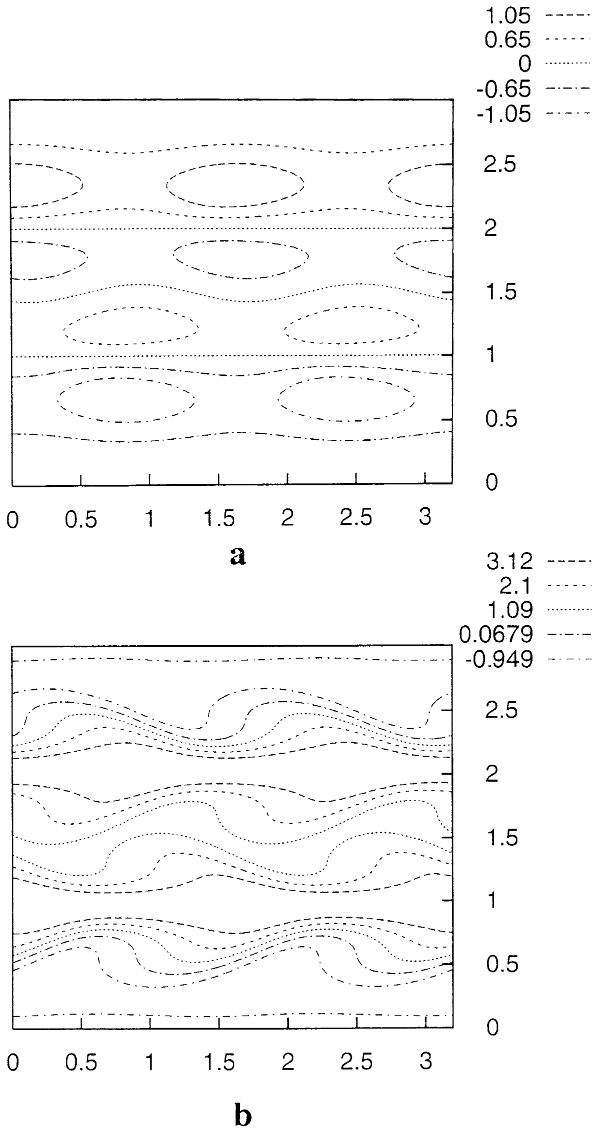

Fig. 2. Snapshot of (a) streamlines and (b) isolines of the temperature deviation for the traveling wave ($M = 1450$, $L = 3.2$).

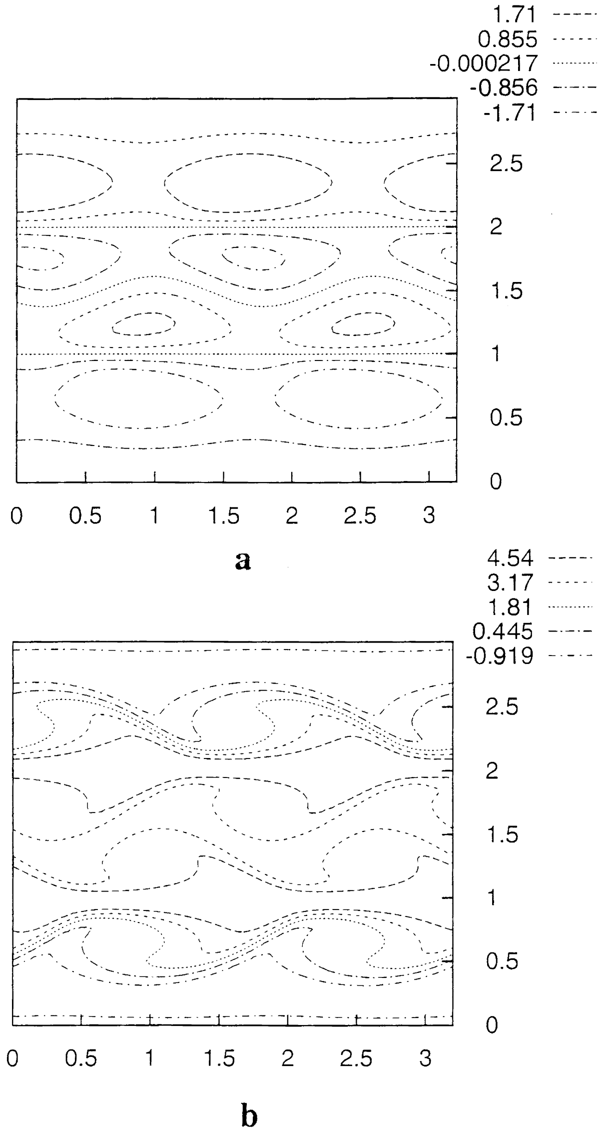

Fig. 3. Snapshot of (a) streamlines and (b) isolines of the temperature deviation for the traveling wave ($M = 2750$, $L = 3.2$).

$m = 1, 2, 3$, corresponding to a parallel flow. For sufficiently large M ($M > 1250$) the parallel flow becomes unstable, and a travelling wave,

$$\psi_m(x, z, t) = \psi_m(x + ct, z), \qquad \Theta_m(x, z, t) = \Theta_m(x + ct, z) \tag{30}$$

is developed. The streamlines and isolines of temperature deviation near the instability threshold are presented in Fig. 2. For any values of M, the wave moves to the *hot end*. This direction of the motion is characteristic for hydrothermal waves [1]. The maximum values of the stream function ($\psi_{\max,m}$) ($m = 1, 2, 3$) are constant in time. All the vortices are positive in the top layer and negative in the bottom layer; convective cells of both signs, forming a chess-order configuration, are developed in the middle layer.

With the increase of the Marangoni number, the intensity of the traveling wave grows (see Fig. 3). The wave velocity c is changed in a non-monotonic way (see Fig. 4). At the larger values of M, the oscillatory instability is developed in the system. The vortices move in the direction of the temperature gradient, changing their form and

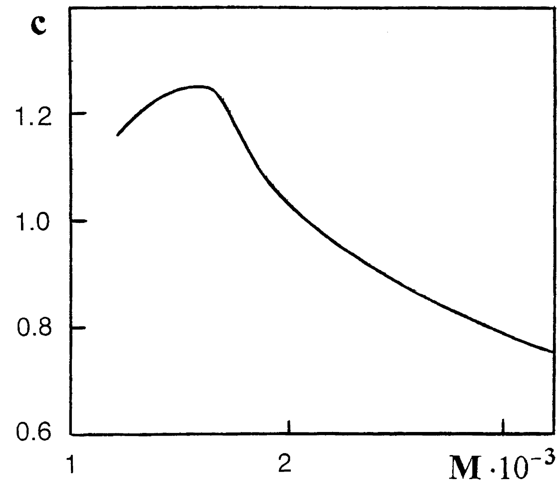

Fig. 4. Dependence of the wave velocity c on the Marangoni number M.

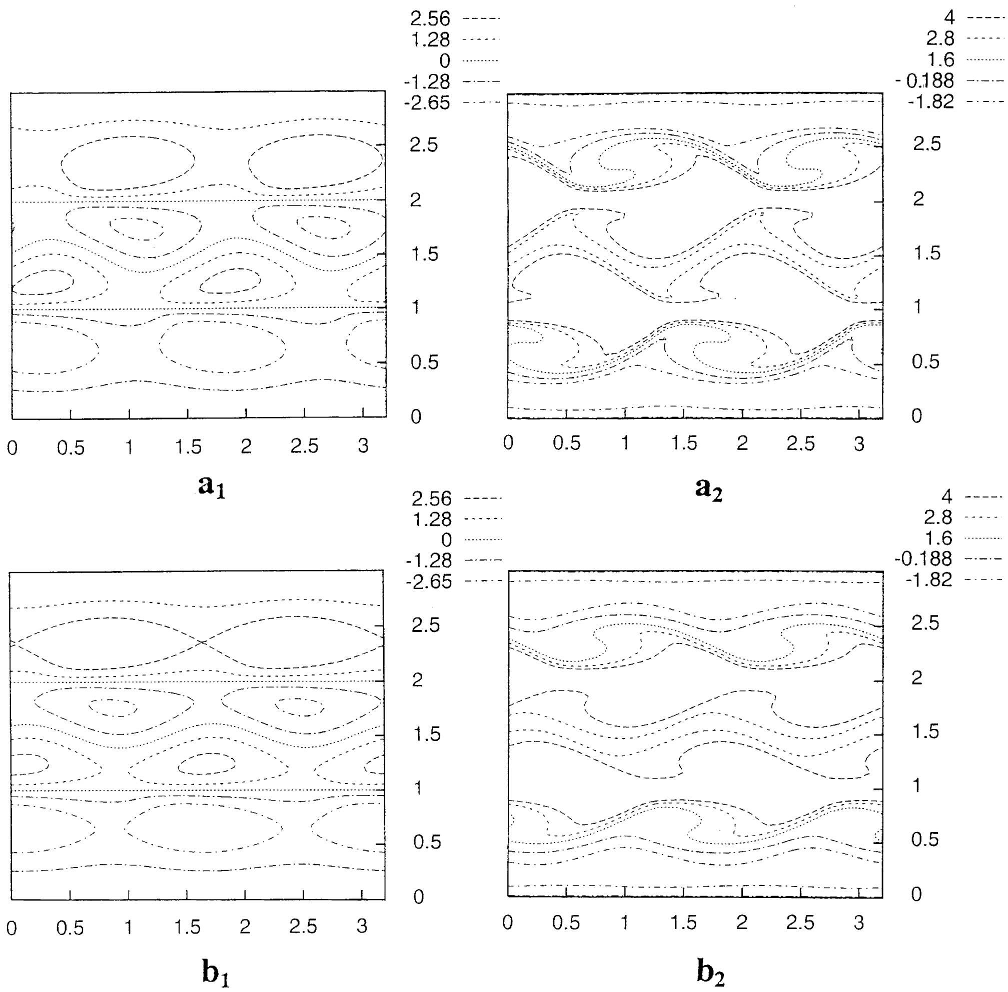

Fig. 5. Snapshots of (a_1, b_1) streamlines and (a_2, b_2) isolines of the temperature deviation for the pulsating traveling wave at different moments of time ($M = 4300$, $L = 3.2$).

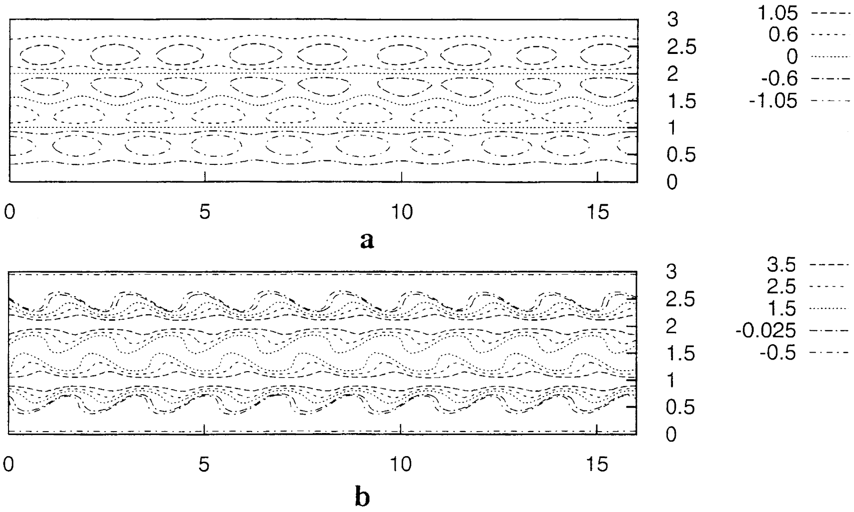

Fig. 6. Snapshot of (a) streamlines and (b) isolines of the temperature deviation for the traveling wave in a long computational region ($M = 1450$, $L = 16$).

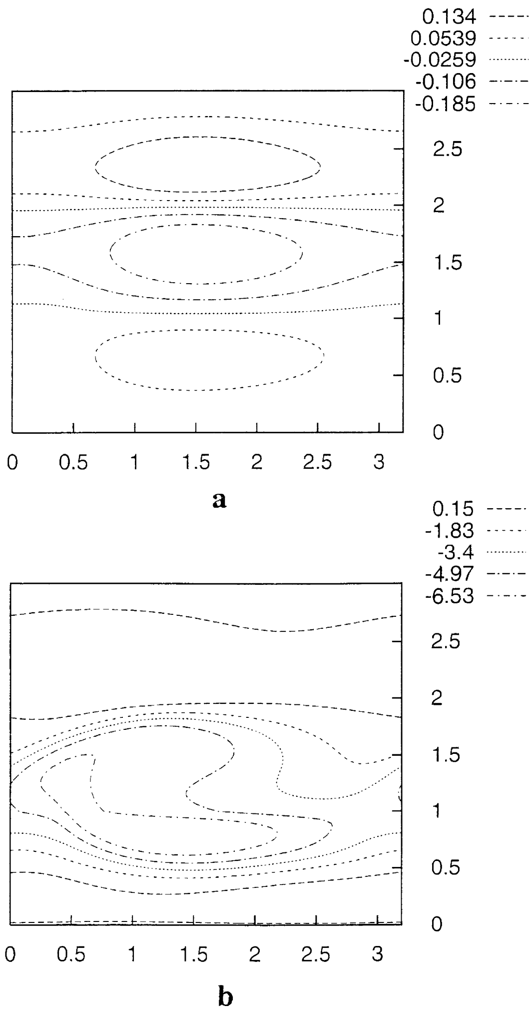

Fig. 7. Snapshot of (a) streamlines and (b) isolines of the temperature deviation for the traveling wave ($M = 2350$, $L = 3.2$).

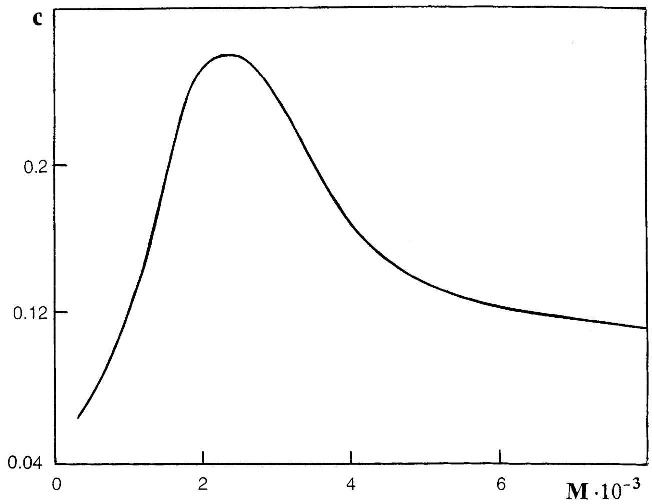

Fig. 8. Dependence of the wave velocity c on the Marangoni number M.

intensity during the oscillatory process (see Fig. 5). It is interesting to note that the waves are excited *in the subcritical way*. Depending on the initial conditions, the system evolves either to the traveling wave or to the parallel flow.

In a long computational region ($L = 16$) the traveling waves described above keep their periodicity (see Fig. 6). Thus, there is no long-wave modulational instability for the thermocapillary traveling waves at the moderate values of M.

4.2. The system air–ethylene glycol–fluorinert FC75

Let us consider the system air–ethylene glycol–fluorinert FC75 with the following set of parameters: $\nu = 0.974$, $\nu_* = 18.767$, $\eta = 0.001$, $\eta_* = 0.013$, $\kappa = 0.098$, $\kappa_* = 0.401$, $\chi = 215.098$, $\chi_* = 606.414$, $\bar{\alpha} = 0.080$. Fix the ratios of the layers thicknesses $a = a_* = 1$.

As in the previous case, for $M \geqslant 450$ the parallel flow becomes unstable, and a traveling wave, moving to the hot end, takes place in the system (see Fig. 7). The vortices are positive in the exterior layers and negative in the middle layer. With the increase of M, the intensity of the motion grows. The dependence of the wave velocity c on the Marangoni number is presented in Fig. 8. The waves are excited in the subcritical way. At the larger values of M ($M \geqslant 8000$) pulsating traveling wave is developed. During these pulsations the maximum values of stream function ($\psi_{\max,m}$) ($m = 1, 2, 3$) are changed in an irregular manner in all the layers.

5. Conclusion

The nonlinear stability of three superposed horizontal liquid layers bounded by two solid planes and subjected to a temperature gradient directed along the interfaces, is investigated. The shape and the amplitude of the convective flows are studied by the finite-difference method. The periodic boundary conditions on the lateral boundaries are considered. The nonlinear simulations of the wavy convective regimes for two real systems, 47v2 silicone oil–water–47v2 silicone oil and air–ethylene glycol–fluorinert FC75, are performed. It is found that the waves move in the direction of the temperature gradient. The wave velocity is changed in a non-monotonic manner. The traveling waves are excited in the subcritical way for both systems. In the long computational region the waves keep their periodicity. Pulsating traveling waves changing their form and intensity, are observed.

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